ALGEBRAIC TRANSFORMATIONS OF A COMPLEX VARIABLE REALIZED BY LINKAGES*

BY

ARNOLD EMCH

1. In the Comptes Rendus† of 1895, Professor KOENIGS has proved the following very interesting theorem:

"*Let $M_1, M_2, \cdots, M_n$ be n points connected by an algebraic relation. This algebraic relation can be realized by a linkage.*"

If $x_1, y_1, z_1; x_2, y_2, z_2; x_3, y_3, z_3; \cdots; x_n, y_n, z_n$ are the rectangular coördinates of the n points, the algebraic relation considered may be written in the form

$$(1) \qquad f(x_1, y_1, z_1, x_2, y_2, z_2, \cdots, x_n, y_n, z_n) = 0.$$

where f is a polynomial in $x_1, y_1, z_1, \cdots, x_n, y_n, z_n$ with real coefficients.

KOENIGS also shows that *the theorem holds when the n points are connected by any number of algebraic relations, provided this number does not make a rigid system of the n points;* the linkage in question is obtained by uniting into one linkage the linkages corresponding respectively to the various relations, in such a way that the points of similar designation in these linkages are spatially identical.

The purpose of this paper is to show how this theorem may be specialized for complex variables.

2. In the first place it is easily seen from KOENIGS's proof that the theorem also holds for the plane, i. e.,

Any number of algebraic relations between n points in a plane can be realized by a plane linkage.

Also in this case the number of relations must be such that the system is movable.

To have a definite idea about the character of the plane linkages to be considered I set down KOENIGS's definition:

* Presented to the Society (Chicago) January 2, 1902. Received for publication March 13, 1902.

† No. 16, p. 861 and no. 18, p. 981. See also

KOENIGS: *Leçons de Cinématique* (1897), pp. 271–308, and in particular p. 302.

KLEIBER: *Beitrag zur kinematischen Theorie der Gelenkmechanismen*, Zeitschrift für Mathematik und Physik, vol. 36 (1891), pp. 296, 328; vol. 41 (1896), pp. 233, 281.

A plane linkage (système articulé plan, Gelenkwerk) is a combination of plates or plane figures subject to remain in one and the same plane, among which a certain number are connected to each other by hinges or pivots perpendicular to the common plane.

In this definition it is assumed that the links move by each other without interference, which means that the links, considered as material, lie in a series of close parallel planes.

Every linkage is constructed in such a manner that one of its pivots is fixed and represents the origin O, while others represent the algebraically related variables. The points of the linkages shall always be designated by the same letters as the corresponding variables.

Two or more linkages each involving two variables may be combined in the following manner: Suppose L, L_1, $L_2 \cdots L_n$ are linkages realizing the transformations

$$u = f(u_1),\ u_1 = f_1(u_2),\ \cdots,\ u_{n-1} = f_{n-1}(u_n),\ u_n = f_n(z).$$

Let the origins of all these linkages coincide; attach the pivot u_n of L_n to the pivot u_n of L_{n-1}; attach the pivot u_{n-1} of L_{n-1} to u_{n-1} of L_{n-2}, and so forth; finally the pivot u_1 of L_1 to u_1 of L. Then, the point u of L evidently realizes the compound transformation

$$u = f\left\{f_1\left[f_2 \cdots f_{n-1}\left(f_n(z)\right)\right]\right\} = F(z). \tag{2}$$

Linkages involving more than two variables may be similarly combined.

The range of effectiveness of a linkage is, of course, limited to a certain finite portion of the plane. This range, although in some cases small, always exists.

3. The proofs of KOENIGS's theorems are based upon the consideration of real quantities. Since, however, an algebraic relation amongst n complex variables $z_1 = x_1 + iy_1, \cdots, z_n = x_n + iy_n$ is equivalent to a pair of real algebraic relations amongst the n coplanar points $(x_1, y_1), \cdots, (x_n, y_n)$, we have the theorem:

Any number of algebraic relations between n complex variables may be realized by a plane linkage.

4. We consider more closely the special case of one algebraic relation

$$f(u, z) = 0 \tag{3}$$

between the complex variables u, z. It is clear that a linkage for the relation

$$w = f(u, z) \tag{4}$$

becomes by fixing w at the origin a linkage for the relation (3), and that a linkage for the relation (4) results from a suitable combination of a number of linkages for the relations

(5) $$z = z_1 + z_2, \qquad z = z_1 z_2.$$

Hence linkages for these fundamental relations (5) have especial interest. Such linkages I proceed to exhibit, the first devised by myself and the second* devised by KLEIBER (loc. cit.).

5. *Addition, Subtraction, Translation.*

Two complex numbers z_1 and z_2 may be added or subtracted by the linkage of Fig. 1, in which all links are equal. Only the point o is fixed. A glance

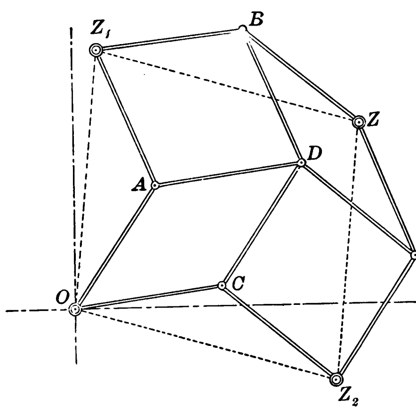

FIG. 1.

at the figure shows that $oz_1 \parallel z_2 z$ and $oz_2 \parallel z_1 z$, no matter how the linkage may be deformed; hence $z = z_1 + z_2$. From the figure it is seen without difficulty that the common range of z_1 and z_2 is a circle having o as a center and twice the length of one link as a radius and that the range of z is a concentric circle with four times the length of one link as a radius. If z_1 and z are given, the point z_2 effects the subtraction $z_2 = z - z_1$. If one of the points, say $z_2 = a$ is constant, then the linkage produces the translation $z = a + z_1$.

6. *Multiplication, Division, Rotation* (KLEIBER).

In order to have a clear understanding of KLEIBER's linkage consider first the "chain" of Fig. 2,* where the shaded triangles are all similar and the remainder of the figure consists of three parallelograms whose connection with the triangles is evident from the figure. It can easily be proved that the triangles

---

* Independently I had devised linkages for $z_1 = az$, $z_1 = z^n$, where a is any complex constant and n is any rational number.

* See KLEIBER, loc. cit., and also W. DYCK's *Katalog mathematischer Modelle*, etc., pp. 318–328.

$A_0D_1A_2$ and $A_1D_2A_3$ and consequently also A_0CA_3 are similar to the shaded triangles. This property holds no matter how the linkage may be distorted.

Combining now two of these linkages with the points A_0, A_1, A_2, A_3 pivoted together, and with the links A_0A_1, A_1A_2, A_2A_3 left out, Fig. 3, and connecting

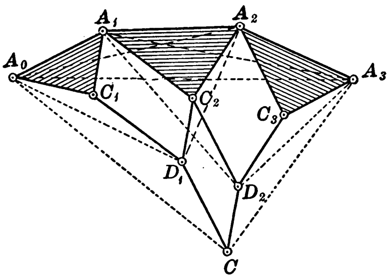

FIG. 2.

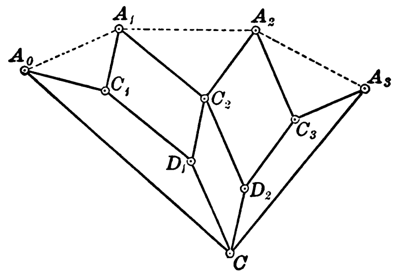

FIG. 3.

A_0C, CA_3, A_3D, DA_0 by links, *so that the quadrilateral* A_0CA_3D *is similar to the quadrilateral* $A_0C_1A_1D_1$, and as a consequence similar to $A_1C_2A_2D_2$ and $A_2C_3A_3D_3$, KLEIBER's linkage, Fig. 4, is obtained. These quadrilaterals remain

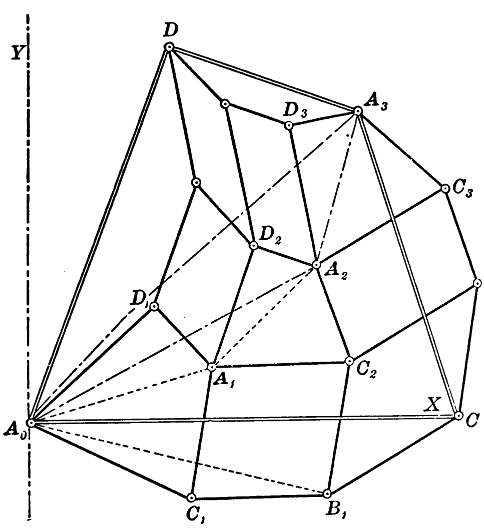

FIG. 4.

similar no matter how the linkage may be distorted. To prove this consider the linkage of Fig. 3, which is the same as that of Fig. 2 with the links A_0A_1, A_1A_2, A_2A_3 left out, but with two new links A_0C and A_3C for which

$$A_0C : A_3C = A_0C_1 : A_1C_1.$$

This linkage when distorted contains the similar triangles of (2) as a special case.

In (3) A_0, A_1, A_2 may be taken arbitrarily, but then A_3 is determined. We make the ratios $A_0A_1 : A_1A_2$, $A_0C_1 : A_1C_2$ equal; then

$$\Delta A_2A_3C \sim \Delta A_0A_1C_1 \sim \Delta A_1A_2C_2 \sim \Delta A_0A_3C.$$

Now in KLEIBER's linkage two linkages (4) X and Y are compounded at the corresponding points A_0, A_1, A_2, A_3. If X and Y have the three points A_0, A_1, A_2 (arbitrarily given) in common, then the A_3's will not coincide. But if A_0, A_1, A_2 are chosen so that $\Delta A_0C_1A \sim \Delta A_1C_2A_2$, then the A_3's of X and Y coincide. This follows from the similitude of the different quadrilaterals and triangles determined by the points A_0, A_1 and A_2. In other words, the compound linkage is movable only when the corresponding triangles and quadrilaterals previously mentioned are similar.

From Fig. 4 it is not difficult to prove the similitude of a number of triangles. Thus, $\Delta A_0A_2A_3 \sim \Delta A_0B_1C$. If A_0C is kept fixed and is assumed as the real unit of a complex plane, with the directions of A_0C and its perpendicular as axes, then B_1 describes a circle with C as a center. Further, A_2 can move in any part of the plane within the range of the linkage and represents the product of the complex variables represented by B_1 and A_3, since $\Delta A_0CB_1 \sim \Delta A_0A_3A_2$ and $A_0C = 1$. In order to remove the restriction that B_1 and A_3 move on circles, in other words, to make the multiplication general, two linkages of the prescribed kind and with corresponding equal links may be pivoted together at their corresponding points A_0, C_1, B_1 and A_0', C_1', B_1'. Owing to the fact that in this compound linkage the triangles $A_0C_1B_1$ and $A_0'C_1'B_1'$ are identical, all the other corresponding triangles of this kind are similar. Thus, the remaining pivots of the second linkage being designated analogously by A_1', A_2', A_3', $\cdots$, two similar triangles $A_0A_2A_3$ and $A_0A_2'A_3'$ are obtained which otherwise are independent of each other.* Now fixing the points A_0, A_2 we take a_0a_2 as the first unit in the real axis of a new complex plane; then in every position of the compound linkage $\Delta A_0A_2A_3 \sim \Delta A_0A_2'A_3'$ and $A_0A_2 = 1$, so that A_3' represents the product of the complex variables represented by A_3 and A_2'. It is evident that by this linkage also the division of two complex numbers may be performed. Rotation and multiplication by a constant are special cases which may easily be arranged from the general linkage. Constructions for limiting positions of the linkage, which the reader may repeat without difficulty, show that the ranges of A_2', A_3' and A_3 are respectively within circles having A_0 as a center,

$$A_0D_1' + D_1'A_1' + A_1'D_2' + D_2'A_2' \qquad \text{and} \qquad A_0D' + D'A_3'$$

* On account of its complexity no separate drawing for the compound linkage has been made. A perfectly clear picture of it is obtained by keeping, in Fig. 4, A_0, C_1, B_1 fixed and giving the linkage a slight displacement. The original linkage and its displacement considered as one give the desired linkage.

as radii, A_2 as a center and $A_2D_3 + D_3A_3'$ as a radius. Fig. 4 is practically the same as the one given by KLEIBER in DYCK's catalogue. As in Fig. 1, which might be constructed with links of two different lengths, the effectiveness of KLEIBER's linkage is increased by taking $A_0D = DA_3 = A_3C = CA_0$, and also all other links equal to one another.

UNIVERSITY OF COLORADO, *March*, 1902.